DISTRIBUTION OF INTEGRAL VALUES FOR THE RATIO OF TWO LINEAR RECURRENCES

CARLO SANNA

ABSTRACT. Let F and G be linear recurrences over a number field $\mathbb{K}$, and let $\mathfrak{R}$ be a finitely generated subring of $\mathbb{K}$. Furthermore, let $\mathcal{N}$ be the set of positive integers n such that $G(n) \neq 0$ and $F(n)/G(n) \in \mathfrak{R}$. Under mild hypothesis, Corvaja and Zannier proved that $\mathcal{N}$ has zero asymptotic density. We prove that $\#(\mathcal{N} \cap [1, x]) \ll x \cdot (\log\log x / \log x)^h$ for all $x \geq 3$, where h is a positive integer that can be computed in terms of F and G. Assuming the Hardy–Littlewood k-tuple conjecture, our result is optimal except for the term $\log \log x$.

1. INTRODUCTION

A sequence of complex numbers $F(n)_{n\in\mathbb{N}}$ is called a *linear recurrence* if there exist some $c_0, \ldots, c_{k-1} \in \mathbb{C}$ $(k \geq 1)$, with $c_0 \neq 0$, such that

$$F(n+k) = \sum_{j=0}^{k-1} c_j F(n+j),$$

for all $n \in \mathbb{N}$. In turn, this is equivalent to an (unique) expression

$$F(n) = \sum_{i=1}^{r} f_i(n)\, \alpha_i^n,$$

for all $n \in \mathbb{N}$, where $f_1, \ldots, f_r \in \mathbb{C}[X]$ are nonzero polynomials and $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ are all the distinct roots of the polynomial

$$X^k - c_{k-1}X^{k-1} - \cdots - c_1 X - c_0.$$

Classically, $\alpha_1, \ldots, \alpha_r$ and k are called the *roots* and the *order* of F, respectively. Furthermore, F is said to be *nondegenerate* if none the ratios α_i/α_j $(i \neq j)$ is a root of unity, and F is said to be *simple* if all the $f_1, \ldots, f_r$ are constant. We refer the reader to [6, Ch. 1–8] for the general theory of linear recurrences.

Hereafter, let F and G be linear recurrences and let $\mathfrak{R}$ be a finitely generated subring of $\mathbb{C}$. Assume also that the roots of F and G together generate a multiplicative torsion-free group. This "torsion-free" hypothesis is not a loss of generality. Indeed, if the group generated by the roots of F and G has torsion order q, then for each $r = 0, 1, \ldots, q-1$ the roots of the linear recurrences $F_r(n) = F(qn + r)$ and $G_r(n) = G(qn + r)$ generate a torsion-free group. Therefore, all the results in the following can be extended just by partitioning $\mathbb{N}$ into the arithmetic progressions of modulo q and by studying each pair of linear recurrences F_r, G_r separately. Finally, define the following set of natural numbers

$$\mathcal{N} := \{n \in \mathbb{N} : G(n) \neq 0,\ F(n)/G(n) \in \mathfrak{R}\}.$$

Regarding the condition $G(n) \neq 0$, note that, by the "torsion-free" hypothesis, $G(n)$ is nondegenerate and hence the Skolem–Mahler–Lech Theorem [6, Theorem 2.1] implies that $G(n) = 0$ only for finitely many $n \in \mathbb{N}$. In the sequel, we shall tacitly disregard such integers.

Divisibility properties of linear recurrences have been studied by several authors. A classical result, conjectured by Pisot and proved by van der Poorten, is the Hadamard-quotient

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Theorem, which states that if $\mathcal{N}$ contains all sufficiently large integers, then F/G is itself a linear recurrence [13, 19].

Corvaja and Zannier [5, Theorem 2] gave the following wide extension of the Hadamard-quotient Theorem (see also [4] for a previous weaker result by the same authors).

Theorem 1.1. *If $\mathcal{N}$ is infinite, then there exists a nonzero polynomial $P \in \mathbb{C}[X]$ such that both the sequences $n \mapsto P(n)F(n)/G(n)$ and $n \mapsto G(n)/P(n)$ are linear recurrences.*

The proof of Theorem 1.1 makes use of the Schmidt's Subspace Theorem. We refer the reader to [3] for a survey on several applications of the Schmidt's Subspace Theorem in Number Theory.

Let $\mathbb{K}$ be a number field. For the sake of simplicity, from now on we shall assume that $\mathfrak{R} \subseteq \mathbb{K}$ and that F and G have coefficients and values in $\mathbb{K}$. Corvaja and Zannier [5, Corollary 2] proved also the following theorem about the set $\mathcal{N}$.

Theorem 1.2. *If F/G is not a linear recurrence, then $\mathcal{N}$ has zero asymptotic density.*

We recall that a set of natural numbers $\mathcal{S}$ has zero asymptotic density if $\#\mathcal{S}(x)/x \to 0$, as $x \to +\infty$, where we define $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$ for all $x \geq 1$.

Corvaja and Zannier also suggested [5, Remark p. 450] that their proof of Theorem 1.2 could be adapted to show that if F/G is not a linear recurrence then

$$\#\mathcal{N}(x) \ll \frac{x}{(\log x)^\delta}, \tag{1}$$

for any $\delta < 1$ and for all sufficiently large $x > 1$, where the implied constant depends on $\mathbb{K}$.

In our main result we obtain a more precise upper bound than (1). Before state it, we mention some special cases of the problem of bounding $\#\mathcal{N}(x)$ that have already been studied.

Alba González, Luca, Pomerance, and Shparlinski [1, Theorem 1.1] proved the following:

Theorem 1.3. *If F is a simple nondegenerate linear recurrence over the integers, $r \geq 2$, $G(n) = n$, and $\mathcal{R} = \mathbb{Z}$, then*

$$\#\mathcal{N}(x) \ll \frac{x}{\log x},$$

for all sufficiently large $x > 1$, where the implied constant depends only on r.

For $G(n) = n$ and $\mathcal{R} = \mathbb{Z}$, a still better upper bound can be given if F is a Lucas sequence, that is, $F(0) = 0$, $F(1) = 1$, and $F(n+2) = aF(n+1) + bF(n)$, for all $n \in \mathbb{N}$ and some fixed integers a and b. In such a case the arithmetic properties of $\mathcal{N}$ were first investigated by André-Jeannin [2] and Somer [16, 17]. Luca and Tron [11] studied the case in which F is the sequence of Fibonacci numbers ($a = b = 1$) and Sanna [15], using some results on the p-adic valuation of Lucas sequences [14], generalized Luca and Tron's result to the following upper bound.

Theorem 1.4. *If F is a nondegenerate Lucas sequences, $G(n) = n$, and $\mathcal{R} = \mathbb{Z}$, then*

$$\#\mathcal{N}(x) \leq x^{1 - \left(\frac{1}{2} + o(1)\right)\frac{\log\log\log x}{\log\log x}},$$

as $x \to +\infty$, where the $o(1)$ depends on F.

Now we state the main result of this paper.

Theorem 1.5. *If F/G is not a linear recurrence, then*

$$\#\mathcal{N}(x) \ll x \cdot \left(\frac{\log\log x}{\log x}\right)^h,$$

for all $x \geq 3$, where h is a positive integer that can be computed in terms of F and G, while the implied constant depends on F and G.

The computation of h will be clear in the proof of Theorem 1.5. In particular, it leads immediately to the following corollary.

Corollary 1.1. *If F/G is not a linear recurrence, $G \in \mathbb{Z}[X]$, and $\gcd(G, f_1, \dots, f_r) = 1$, then h can be taken as the number of irreducible factors of G in $\mathbb{Z}[X]$ (counted without multiplicity).*

Except for the term $\log\log x$, Corollary 1.1 should be optimal. Indeed, pick a positive integer h and an *admissible* h-tuple $\mathbf{h} = (n_1, \dots, n_h)$, that is, $n_1 < \cdots < n_h$ are positive integers such that for each prime number p there exists a residue class modulo p which does not intersect $\{n_1, \dots, n_h\}$. Assuming Hardy–Littlewood h-tuple conjecture [7, p. 61], we have that the number $T_{\mathbf{h}}(x)$ of positive integers $n \leq x$ such that $n + n_1, \dots, n + n_h$ are all prime numbers satisfies

$$T_{\mathbf{h}}(x) \sim C_{\mathbf{h}} \cdot \frac{x}{(\log x)^h},$$

as $x \to +\infty$, where $C_{\mathbf{h}} > 0$ depends on $\mathbf{h}$. Therefore, taking $F(n) = (2^{n+n_1} - 2) \cdots (2^{n+n_h} - 2)$ and $G(n) = (n + n_1) \cdots (n + n_h)$, we obtain

$$\#\mathcal{N}(x) \geq T_{\mathbf{h}}(x) \gg \frac{x}{(\log x)^h},$$

for all sufficiently large $x > 1$.

Notation. Hereafter, the letter p always denotes a prime number. We employ the Landau–Bachmann "Big Oh" and "little oh" notations O and o, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. If $A \ll B$ and $A \gg B$, we write $A \asymp B$. Any dependence of implied constants is explicitly stated or indicated with subscripts.

2. Preliminaries

First, we need a quantitative form of a result due to Kronecker [10] (see also [18, p. 32]), which states that the average number of zeros modulo p of a nonconstant polynomial $f \in \mathbb{Z}[X]$ is equal to the number of irreducible factors of f in $\mathbb{Z}[X]$.

Theorem 2.1. *Given a nonconstant polynomial $f \in \mathbb{Z}[X]$, for each prime number p let $\eta_f(p)$ be the number of zeros of f modulo p. Then*

$$\sum_{p \leq x} \eta_f(p) \cdot \frac{\log p}{p} = h \log x + O_f(1),$$

for all $x \geq 1$, where h is the number of irreducible factors of f in $\mathbb{Z}[X]$.

Proof. It is enough to prove the claim for irreducible f. Let $\mathcal{G}$ be the Galois group of f over $\mathbb{Q}$. By a quantitative version of the Chebotarev's density theorem [12, Ch. 2, Theorem 7.2], the number of primes $p \leq x$ such that the irreducible factors of f modulo p have degrees $d_1, \dots, d_s$ is

$$\frac{\pi_{\mathcal{G}}(d_1, \dots, d_s)}{\#\mathcal{G}} \cdot \operatorname{Li}(x) + O_f\left(\frac{x}{\exp(C\sqrt{\log x})}\right),$$

for all $x > 1$, where $\operatorname{Li}(x)$ is the logarithmic integral function, $C > 0$ is a constant depending on f, and $\pi_{\mathcal{G}}(d_1, \dots, d_s)$ is the number of $g \in \mathcal{G}$ that have cycle decomposition with lengths $d_1, \dots, d_s$ when regarded as permutations of the roots of f. Furthermore, $\mathcal{G}$ acts transitively on the roots of f, since f is irreducible, hence

$$\sum_{g \in \mathcal{G}} \#X^g = \#\mathcal{G},$$

by Burnside's lemma, where X^g is the set of roots of f which are fixed by g. Hence,

$$\sum_{p \leq x} \eta_f(p) = \operatorname{Li}(x) + O_f\left(\frac{x}{\exp(C\sqrt{\log x})}\right),$$

and the desired result follows by partial summation. □

The following lemma [5, Lemma A.2] regards the minimum of the multiplicative orders of some fixed algebraic numbers modulo a prime ideal.

Lemma 2.2. *Let $\beta_1, \dots, \beta_s \in \mathbb{K}$ such that none of them is zero or a root of unity. Then, for all $x \geq 1$, the number of prime numbers $p \leq x$ such that some β_i has order less than $p^{1/4}$ modulo some prime ideal of $\mathcal{O}_{\mathbb{K}}$ lying above p is $O(x^{1/2})$, where the implied constant depends only on $\beta_1, \dots, \beta_s$.*

Now we state a technical lemma about the cardinality of a sieved set of integers.

Lemma 2.3. *For each prime number p, let $\Omega_p \subsetneq \{0, 1, \dots, p-1\}$ be a set of residues modulo p, and denote by Ω the whole family of Ω_p's. Suppose that there exist constants $c, h > 0$ such that $\#\Omega_p \leq c$ for each prime number p and*

$$\sum_{p \leq x} \#\Omega_p \cdot \frac{\log p}{p} = h \log x + O(1), \tag{2}$$

for all $x > 1$. Then we have

$$\#\left\{n \leq x : (n \bmod p) \notin \Omega_p,\ \forall p \in \left]y, z\right]\right\} \ll_{\Omega,\delta_1,\delta_2} x \cdot \left(\frac{\log y}{\log x}\right)^h,$$

for all $\delta_1, \delta_2 > 0$, $x > 1$, $2 \leq y \leq (\log x)^{\delta_1}$, and $z \geq x^{\delta_2}$.

Proof. All the constants in this proof, included the implied ones, may depend on Ω, δ_1, δ_2. Clearly, we can assume $\delta_2 \leq 1/2$. By the large sieve inequality [8, Theorem 7.14], we have

$$\#\left\{n \leq x : (n \bmod p) \notin \Omega_p,\ \forall p \in \left]y, z\right]\right\} \ll x \cdot \left(\sum_{m \leq w} g_y(m)\right)^{-1}, \tag{3}$$

where $w := x^{\delta_2}$ and g_y is the multiplicative arithmetic function supported on squarefree numbers with all prime factors $> y$ and such that

$$g_y(p) = \frac{\#\Omega_p}{p - \#\Omega_p},$$

for any prime number $p > y$.

For sufficiently large x, we have $y \leq w$, and it follows from (2) that

$$-(A + h \log y) + h \log w \leq \sum_{p \leq w} g_y(p) \log p \leq B + h \log w,$$

for some constants $A, B > 0$. Then from [9, Theorem 0.4.1] we obtain that

$$\sum_{m \leq w} g_y(m) = \frac{\mathfrak{S}(w)}{\Gamma(h+1)} \cdot (\log w)^h \cdot \left(1 + O\left(\frac{\log y}{\log w}\right)\right),$$

where Γ is the Euler's Gamma function and

$$\mathfrak{S}(w) := \prod_{p \leq w} (1 + g_y(p)) \left(1 - \frac{1}{p}\right)^h.$$

In particular, since $y \leq (\log x)^{\delta_1}$, for sufficiently large x we get that

$$\sum_{m \leq w} g_y(m) \gg \mathfrak{S}(w) \cdot (\log w)^h. \tag{4}$$

Now from (2) it follows easily that

$$\prod_{p \leq t} \left(1 - \frac{\#\Omega_p}{p}\right)^{-1} \asymp (\log t)^h,$$

for all $t \geq 2$. Hence, also thanks to Mertens' third theorem [8, p. 34, Eq. 2.16], we have

$$\mathfrak{S}(w) = \prod_{p \leq w} \left(1 - \frac{\#\Omega_p}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^h \Big/ \prod_{p \leq y} \left(1 - \frac{\#\Omega_p}{p}\right)^{-1} \gg \frac{1}{(\log y)^h}. \tag{5}$$

Putting together (3), (4), and (5), and recalling that $w = x^{\delta_2}$, the desired result follows. $\square$

Finally, we need a lemma about the number of zeros of a simple linear recurrence in a finite field of q elements $\mathbb{F}_q$ (see also [6, Theorem 5.10] for a more precise result).

Lemma 2.4. *Let $c_1, \ldots, c_r, a_1, \ldots, a_r \in \mathbb{F}_q^*$, and let N be the minimum of the orders of the a_i/a_j $(i \neq j)$ in $\mathbb{F}_q^*$. (If $r = 1$ then pick an arbitrary positive integer N.) Then the number of integers $m \in [0, q-1]$ such that*

$$\sum_{i=1}^{r} c_i a_i^m = 0 \tag{6}$$

is at most $5(q-1)N^{-1/2^{r-2}}$.

Proof. For $r = 1$ the claim is obvious since (6) never holds, hence we may assume $r \geq 2$. In [5, Proposition A.1] it is stated and proved that for prime q the number of integers $m \in [1, q-1]$ satisfying (6) is at most $4(q-1)N^{-1/2^{r-2}}$, and the same proof works also for not necessarily prime q. Thus the claim follows, since $4(q-1)N^{-1/2^{r-2}} + 1 \leq 5(q-1)N^{-1/2^{r-2}}$. $\square$

3. Proof of Theorem 1.5

The first part of the proof proceeds similarly to the proof of Theorem 1.2. If $\mathcal{N}$ is finite, then the claim is trivial, hence we suppose that $\mathcal{N}$ is infinite. Then, by Theorem 1.1 it follows that $F/G = H/P$, for some linear recurrence H and some polynomial P. As a consequence, without loss of generality, we shall assume that G is a polynomial.

Let S be a finite set of absolute values of $\mathbb{K}$ containing all the archimedean ones. Write $\mathcal{O}_S$ for the ring of S-integers of $\mathbb{K}$, that is, the set of all $\alpha \in \mathbb{K}$ such that $|\alpha|_v \leq 1$ for all $v \notin S$. Enlarging $\mathbb{K}$ and S we may assume that $\alpha_1, \ldots, \alpha_r$ are S-units, $f_1, \ldots, f_r, G \in \mathcal{O}_S[X]$, and $\mathfrak{R} \subseteq \mathcal{O}_S$.

Since F/G is not a linear recurrence, it follows that G does not divide all the $f_1, \ldots, f_r$. Moreover, factoring out the greatest common divisor $(G, f_1, \ldots, f_r)$ we can even assume that $(G, f_1, \ldots, f_r) = 1$ and that G is nonconstant. In particular, $(G(n), f_1(n), \ldots, f_r(n))$ is bounded and, enlarging S, we may assume that it is an S-unit for all $n \in \mathbb{N}$.

Let $N_{\mathbb{K}}(\alpha)$ denotes the norm of $\alpha \in \mathbb{K}$ over $\mathbb{Q}$. It is easy to prove that there exist a positive integer g and a nonconstant polynomial $\widetilde{G} \in \mathbb{Z}[X]$ such that $N_{\mathbb{K}}(G(n)) = \widetilde{G}(n)/g$ for all $n \in \mathbb{N}$. Let h be the number of irreducible factors of $\widetilde{G}$ in $\mathbb{Z}[X]$. Again by enlarging S, we may assume that g is an S-unit.

Let $\mathcal{P}$ be the set of all prime numbers p which do not make $\widetilde{G}$ vanish identically modulo p, such that $p\mathcal{O}_{\mathbb{K}}$ has no prime ideal factor π_v with $v \in S$, and such that the minimum order of the α_i/α_j $(i \neq j)$ modulo any prime ideal above p is at least $p^{1/4}$. Furthermore, let us define

$$\Omega_p := \left\{\ell \in \{0, \ldots, p-1\} : \widetilde{G}(\ell) \equiv 0 \pmod{p}\right\},$$

for any $p \in \mathcal{P}$, and $\Omega_p := \varnothing$ for any prime number $p \notin \mathcal{P}$.

Let $x \geq 3$, $y := (\log x)^{2^r h}$, and $z := x^{1/(d+1)}$, where $d := [\mathbb{K} : \mathbb{Q}]$. We split $\mathcal{N}(x)$ into two subsets:

$$\begin{aligned} \mathcal{N}_1 &:= \{n \in \mathcal{N}(x) : (n \bmod p) \notin \Omega_p,\ \forall p \in \,]y, z]\}, \\ \mathcal{N}_2 &:= \mathcal{N} \setminus \mathcal{N}_1. \end{aligned}$$

First, we give an upper bound for $\#\mathcal{N}_1$. Hereafter, all the implied constants may depend on F and G. Clearly, $\#\Omega_p \subsetneq \{0, 1, \ldots, p-1\}$ and $\#\Omega_p \leq \deg(\widetilde{G})$ for all prime number p, while from Theorem 2.1 and Lemma 2.2 it follows that

$$\sum_{p \leq x} \#\Omega_p \cdot \frac{\log p}{p} = h \log x + O(1).$$

Therefore, applying Lemma 2.3, we obtain

$$\#\mathcal{N}_1 \ll x \cdot \left(\frac{\log y}{\log x}\right)^h \ll \left(\frac{\log\log x}{\log x}\right)^h.$$

Now we give an upper bound for $\#\mathcal{N}_2$. If $n \in \mathcal{N}_2$ then there exist $p \in \mathcal{P} \cap \,]y,z]$ and $\ell \in \Omega_p$ such that $n \equiv \ell \pmod p$. In particular, p divides $N_{\mathbb{K}}(G(\ell))$ in $\mathcal{O}_S$ and, since $p\mathcal{O}_{\mathbb{K}}$ has no prime ideal factor π_v with $v \in S$, it follows that there exists some prime ideal π of $\mathcal{O}_S$ lying above p and dividing $G(\ell)$. Let $\mathbb{F}_q := \mathcal{O}_S/\pi$, so that q is a power of p. Write $n = \ell + mp$, for some integer $m \geq 0$. Since π divides $G(n)$ and $F(n)/G(n) \in \mathcal{O}_S$, we have that $F(n)$ is divisible by π too. As a consequence, we obtain that

$$\sum_{i=1}^r f_i(\ell)\,\alpha_i^\ell\left(\alpha_i^p\right)^m \equiv \sum_{i=1}^r f_i(n)\,\alpha_i^n \equiv F(n) \equiv 0 \pmod \pi. \tag{7}$$

Note that $f_1(\ell), \dots, f_r(\ell)$ cannot be all equal to zero modulo π, since π divides $G(\ell)$ and $(G(\ell), f_1(\ell), \dots, f_r(\ell))$ is an S-unit. Note also that the minimum order of the α_i^p/α_j^p $(i \neq j)$ modulo π is equal to the minimum order of the α_i/α_j $(i \neq j)$ modulo π, since $(p, q-1) = 1$.

Therefore, we can apply Lemma 2.4 to the congruence (7). The positive integer r may decrease, and N can the taken $\geq p^{1/4}$, in light of the definition of $\mathcal{P}$. It follows that the number of possible values of m modulo $q-1$ is at most $5(q-1)p^{-1/2^r}$. Consequently, the number of possible values of $n \leq x$ is at most

$$5(q-1)p^{-1/2^r}\left(\frac{x}{p(q-1)} + 1\right) \ll \frac{x}{p^{1+1/2^r}},$$

since $p(q-1) < p^{d+1} \leq z^{d+1} \leq x$. Hence, we have

$$\#\mathcal{N}_2 \ll \sum_{p \in \mathcal{P} \cap\,]y,z]} \frac{x}{p^{1+1/2^r}} \ll \int_y^{+\infty} \frac{\mathrm{d}t}{t^{1+1/2^r}} \ll \frac{x}{y^{1/2^r}} = \frac{x}{(\log x)^h}.$$

In conclusion,

$$\#\mathcal{N}(x) = \#\mathcal{N}_1 + \#\mathcal{N}_2 \ll x \cdot \left(\frac{\log\log x}{\log x}\right)^h$$

as claimed.

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Università degli Studi di Torino, Department of Mathematics, Turin, Italy
E-mail address: carlo.sanna.dev@gmail.com